# Relativistic Physics of GPS

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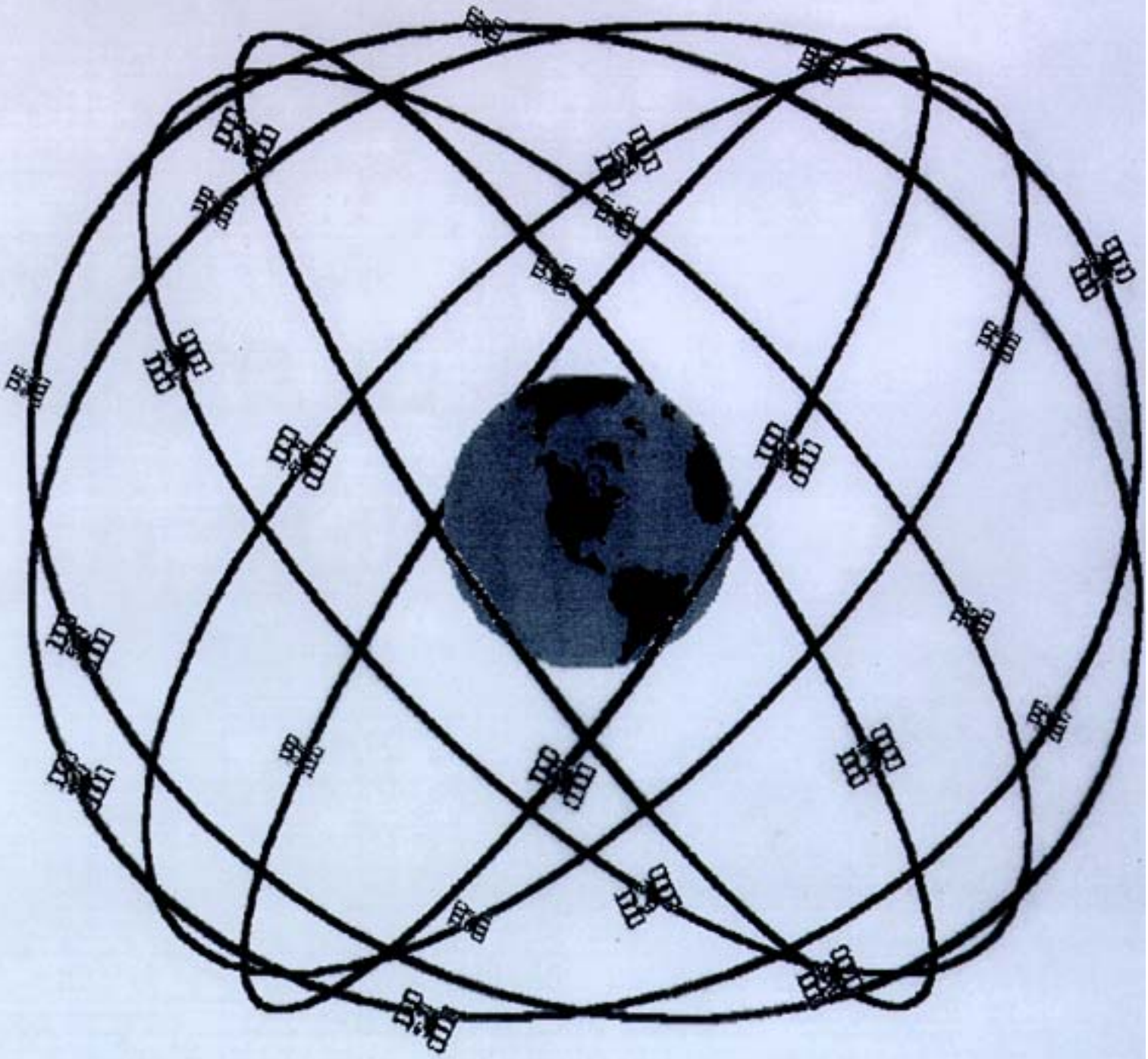
# Outline

- GPS - the system
- Signals broadcast by GPS
- GPS error sources & Atmospheric Effects
- Relativistic Effects in GPS
- Relativity Physics of GPS Receivers
- Navigation & Time Transfer
  - Minkowskii Space-Time
  - Curved Space-Time
- Clock Synchronization
  - Special Relativity
  - General Relativity
- Sagnac Effect
- Strange Stuff in GPS Time Transfer Data



**GPS Nominal Constellation**
**24 Satellites in 6 Orbital Planes**
**4 Satellites in each Plane**
**20,200 km Altitudes, 55 Degree Inclination**

From: Peter H. Dana, The Geographer's Craft Project, Department of Geography, The University of Texas at Austin. Copyright © 1998 Peter H. Dana

# GPS Overview

## Brief History

Predecessors to GPS: Transit (operational, NAVY, APL)
Timation (experimental, NAVY, NRL)
621B (AF, Aerospace Corp., study program)

1973 GPS program approved, JPO established (El Segundo, CA)

1978 AF begins launches of experimental (Block I) satellites
→ 11 launched, one failure

1989 First launch of Block II satellites
- 4 Atomic Clocks: 2 Cs and 2 Rb
- designed to be fully operational
- radiation hard electronics
- Selective Availability (SA) and Anti-Spoofing (A-S

1995 GPS Full Operational Capability announced

## Current Constellation

- 24 Satellites in 6 orbital planes, inclined at 55° w.r.t. equa
- ≈ circular orbits, semi-major axis ≈ 26,562 km
- Orbital periods ≈ $\frac{1}{2}$ sidereal day ≈ 12 hours
  SV passes over same place on earth every ≈ 23 hours 56 min
- At least 4 satellites in view anywhere on earth

Navigation Signals Pseudorandom noise (PRN) Code
Effectively Provides: satellite position and time of transmission event

Navigation Codes

| | Code | Length | Period |
|---|---|---|---|
| civilian | C/A | 1023 bits | 1 ms |
| DoD | P<br>Y | $6.18 \times 10^{12}$ bits | 7 days |

RF Signals

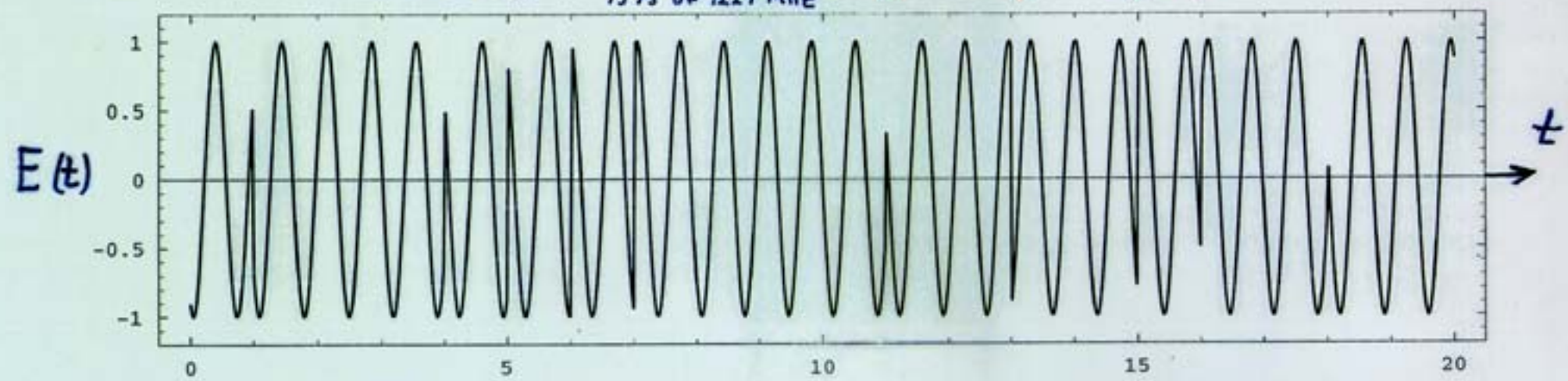
| Navigation Bands | Frequency (MHz) | Code |
|---|---|---|
| $L_1$ | 1575.42 | C/A, P(Y) |
| $L_2$ | 1227.6 | P(Y) |

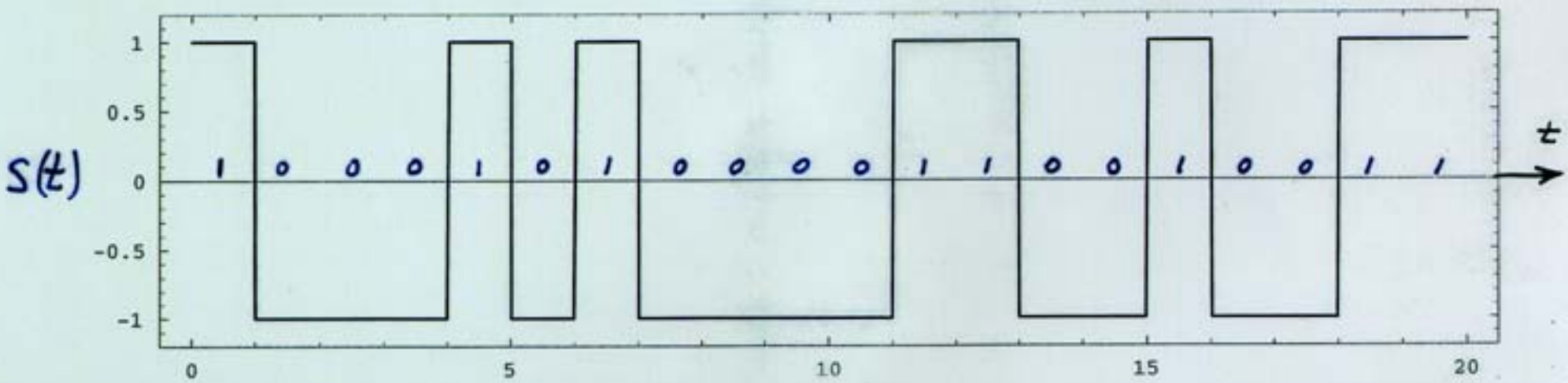
# GPS Signal : Phase Modulation

$$E(t) = \underbrace{S(t)}_{\substack{+1,\,-1 \\ 10.23\ \text{MHz}}} \cos(\omega_0 t)$$

carrier frequency
1575 or 1227 MHz

Phase discontinuities form invariant 3-d hypersurfaces in space-time



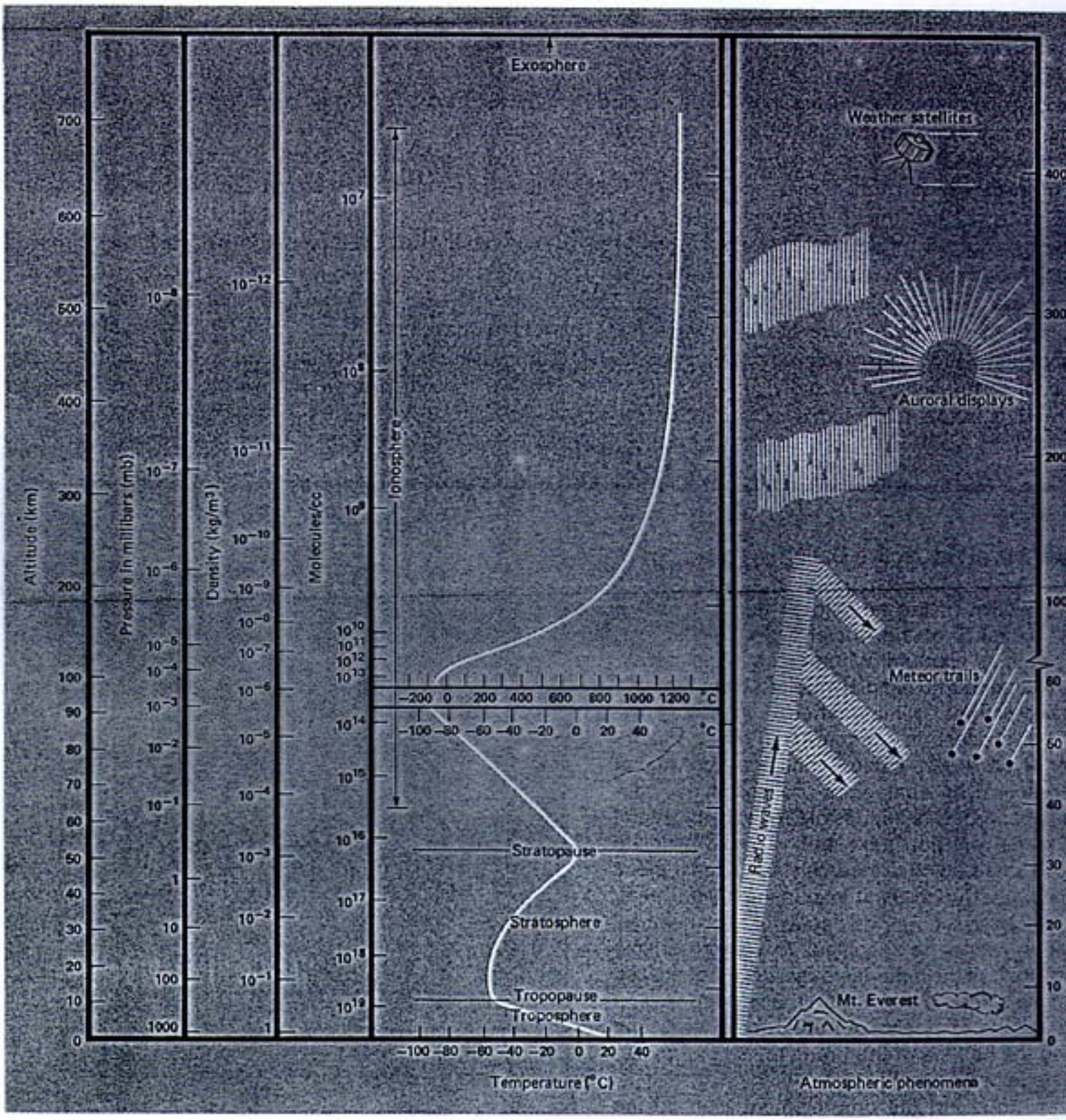



**Figure 9.5** A cross section of the structure of the earth's atmosphere.

From: M. Zeilik, Astronomy: The Evolving Universe, Harper and Row, 1979.

# Physical Effects in GPS

→ Actual signal is measured in frame of reference of moving receiver – not ECI frame.

## Range Errors (PPS Receiver)

Ionospheric delay (100 – 700 km above earth surface) – depends on TEC due to ionized O and N
– $\lesssim$ 1 cm, residual based on $L_1$ and $L_2$ frequency delay

Tropospheric delay – depends on atmospheric pressure and humidity
– ~ 0.7 m (after correction with model)

Clock and Ephemeris Error ~ 3.6 m
Multi-path reflection error ~ 1.8 m
Receiver Noise ~ 0.6 m

User Equivalent Range Error (UERE) $\approx$ 4.1 m $(1-\sigma)$ [UERE is given by square root of sum of squares]

Typical Horizontal Dilution of Precision (HDOP) $\approx$ 2.0

Total Stand-Alone Horizontal Accuracy = 2 × (UERE) × (HDOP)
= 16.4 m (2 drms)

Definition: $2\,\text{drms} = 2\left(\sigma_N^2 + \sigma_E^2\right)$

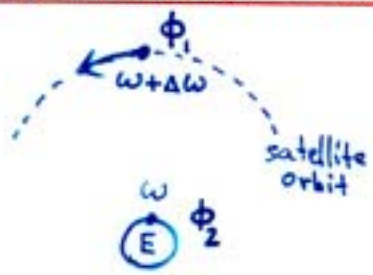
98% of values contained within 2 drms radius
(Gaussian 2-d distribution)

# Relativistic Effects in GPS

Space-Time is curved: $ds^2 = g_{ij}\, dx^i dx^j$, $i,j = 0,1,2,3$

**Scales:** Atomic clock stability is 1 part in $10^{13}$ or $10^{14}$.
Light Travel 1 ns $\cong$ 30 cm $\sim$ 1 foot

## Gravitational Redshift




Difference in gravitational potential between satellite orbit and earth causes satellite clocks to run fast by 45 $\mu$s/day.

$$\frac{\Delta\omega}{\omega} = \frac{\phi_1 - \phi_2}{c^2}, \qquad \phi = -\frac{GM}{r}$$

$$\approx 5.28 \times 10^{-10}$$

## Time Dilation

$$\Delta t' = \frac{\Delta t}{\gamma}, \quad \frac{1}{\gamma} \cong 1 + \underbrace{\frac{v^2}{2c^2}}_{8.33\times10^{-11}}, \quad v \cong 8.37\ \frac{\text{km}}{\text{s}}$$

Correction is 1000 times larger than minimal atomic clock stability of 1 part in $10^{13}$.

$\Rightarrow$ Causes satellite clocks to run slow by 7 $\frac{\mu s}{day}$.

## Net Effect of Redshift & Time Dilation

$\Rightarrow$ Satellite clocks run fast 38 $\mu$s/day.

Compensation: Satellite clocks are "artificially" slowed down

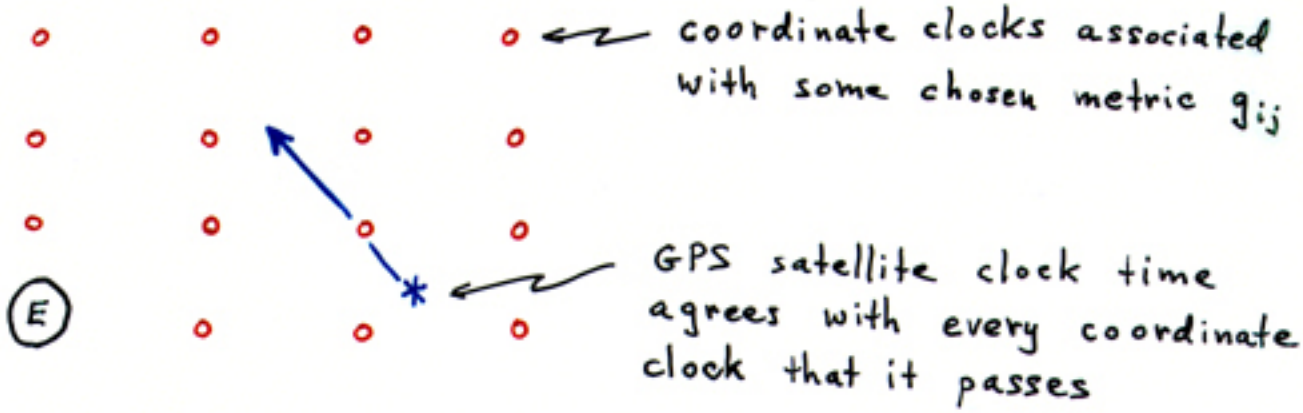
$\frac{\Delta f}{f} \cong -4.46 \times 10^{-10}$ frequency offset applied before launch

## Effect of Orbit Eccentricity on Coordinate Time of Transmission

"e sin E" correction – applied by receiver

$$\Delta t_r = \frac{2}{c^2}\sqrt{GMa}\ \underset{\substack{\uparrow \\ \text{eccentricity}}}{e}\ \sin \underset{\substack{\uparrow \\ \text{eccentric anomaly}}}{E} \sim 23\ \text{ns} \quad \text{for } e \sim 0.01$$

# What time do GPS Satellite Clocks Keep?




Real clocks keep proper time

$$d\tau = \frac{ds}{c} = \left( g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} \right)^{1/2} dt$$

relation between proper time $\tau$ and coordinate time $x^0 = ct$ is given by world line $x^i(t)$

GPS
KEY IDEA

**GPS satellite clocks keep coordinate time of a specific metric $g_{ij}$**

(satellite can broadcast the emission event coordinates of it's signal)

Achieved approximately in practice by "factory offset" in GPS satellite clock plus "e sin E" correction made in receiver.

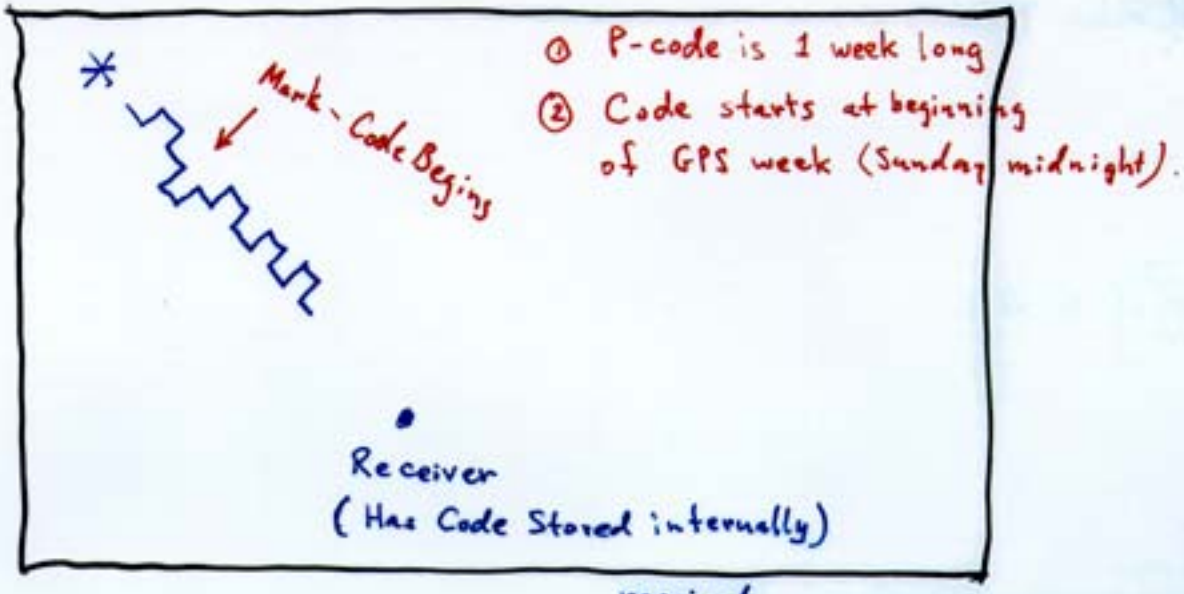
COMMENT: Relation $\underset{\text{proper time}}{\tau} \leftrightarrow \underset{\text{coord. time}}{t}$ is path dependent

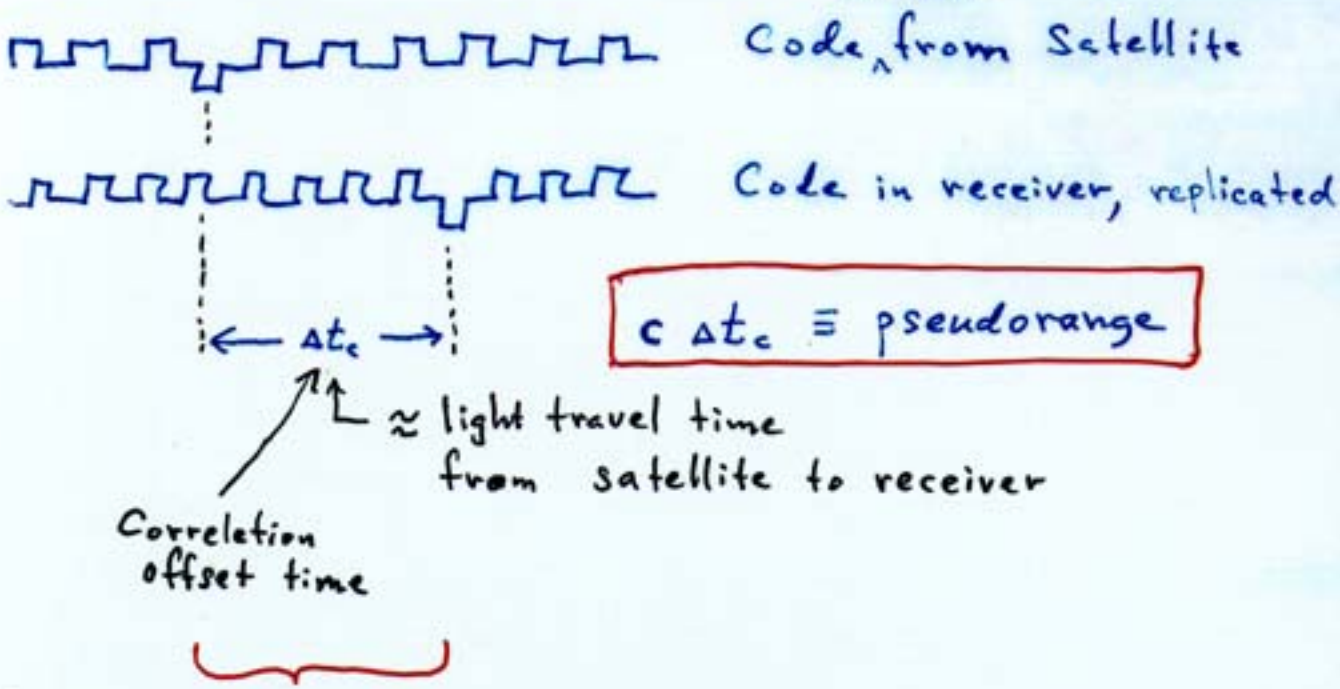
(a) $\tau = f(t)$ is in general not a simple constant:
$$\tau = k\,t$$

(b) Master control station must predict $\tau = f(t)$ into the future, i.e., what the expected world line will be.

$$c\,\Delta t_c = \text{Pseudorange}$$

$$c\,(\text{Correletion Offset}) = \text{Pseudorange}$$

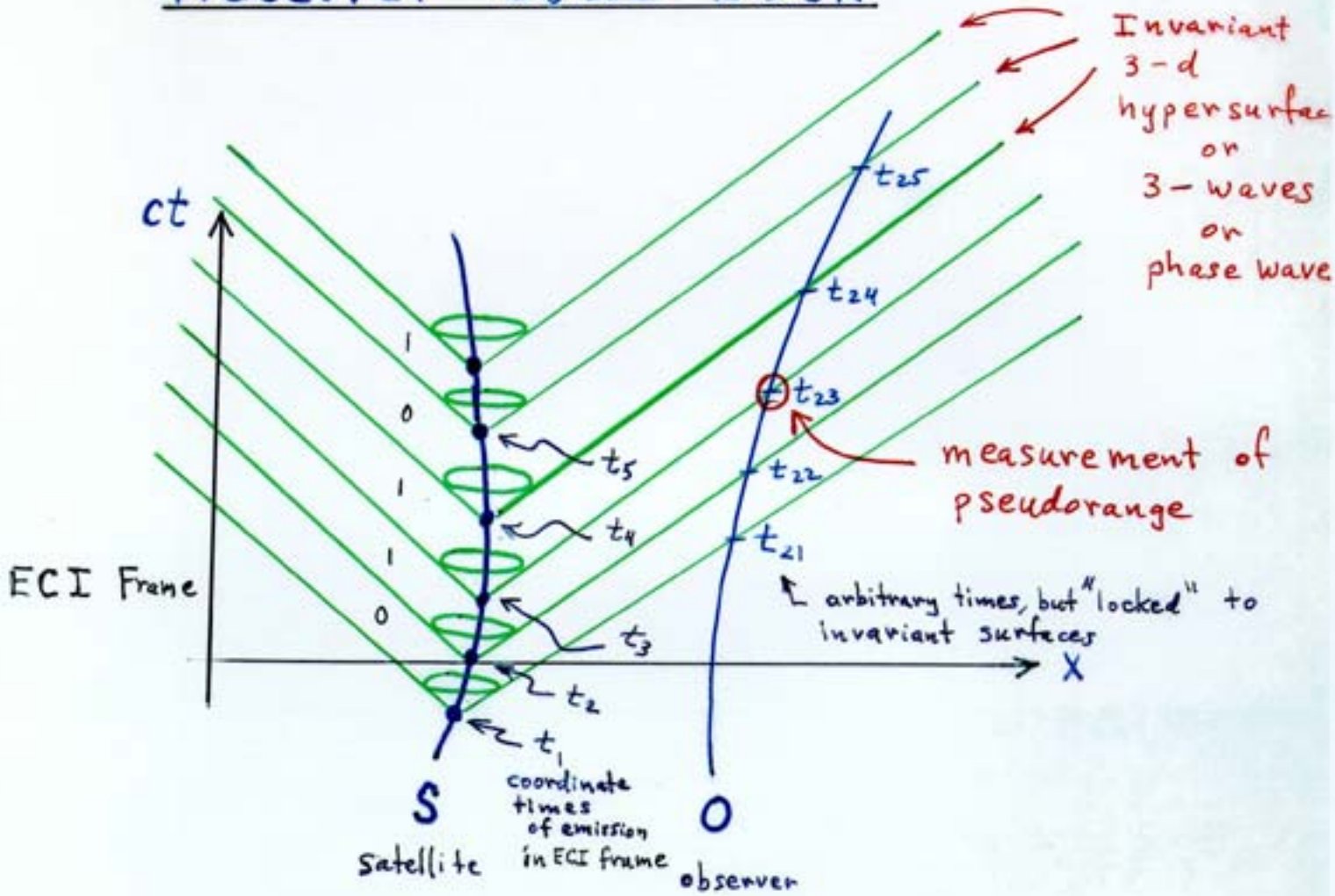



$$\boxed{c\,\Delta t_c \equiv \text{pseudorange}}$$

Phase difference between code received from satellite and code replicated by receiver is an invariant (to within a constant) under Lorentz transformations.

# Receiver Code Lock




Receiver (observer) correlates replicated code with code received from the satellite.

$$\text{Measured pseudorange} = c\,(t_{23} - t_3) = \text{phase difference}$$

$c$: constant

$t_{23}$: receiver phase

$t_3$: invariant phase of 3-wave

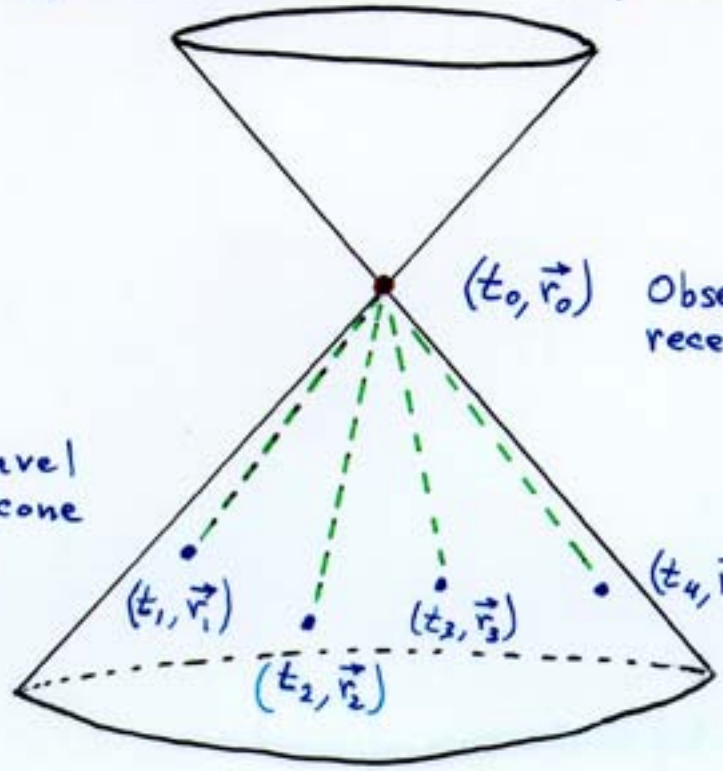
phase difference: relativistic invariant between replicated code in receiver and code received from satellite

To achieve code lock, the receiver must search in $\underbrace{\text{frequency}}$ and $\underbrace{\text{time}}$.

frequency: rate of received code bits in receiver comoving frame varies with receiver velocity

time: code offset due to propagation time plus arbitrary phase of replicated code.

# Navigation in Flat Space-Time




$$(\vec{r}_0 - \vec{r}_n)^2 = c^2 [t_0 - t_n]^2$$

unknown observer position

reception time on receiver (hardware) clock

unknow reception time $= \tau^* + \underbrace{\delta\tau^*}_{\text{receiver clock bias}}$

$$= \Big[ \underbrace{c(\tau^* - t_n)}_{\rho_n = \text{measured PRN code phase difference (pseudorange)}} + \delta\tau^* \Big]^2$$

$$\rho_n = \text{measured PRN code phase difference (pseudorange)} = c\,\frac{\Delta\Phi}{f_0}$$

known from broadcast ephemeris

$$(\vec{r}_0 - \vec{r}_n)^2 = \Big[ \underbrace{\rho_n}_{\text{measured pseudorange a relativistic invariant}} + \delta\tau^* \Big]^2 , \quad n = 1, 2, 3, 4 \text{ satellites}$$

3 unknowns

1 unknown

measured pseudorange a relativistic invariant

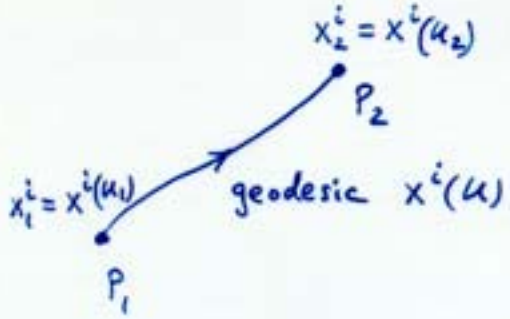
## Time Transfer

Observer position $\vec{r}_0$ is known. One satellite (only measurement $\rho_1$ is needed.

# Theory of Navigation in Curved Space-Time

## Covariant Navigation Equations

T. B. Bahder
Am. J. Phys. 2001
(in press)




$$\frac{1}{2}(\text{space-time distance})^2 = \Omega(x_1^i, x_2^j) = \frac{1}{2}(u_2 - u_1)\int_{u_1}^{u_2} g_{ij}\,\frac{dx^i}{du}\frac{dx^j}{du}\,du$$

<u>World Function</u>
or <u>Eikonal</u> (Optics): J. L. Synge 1960

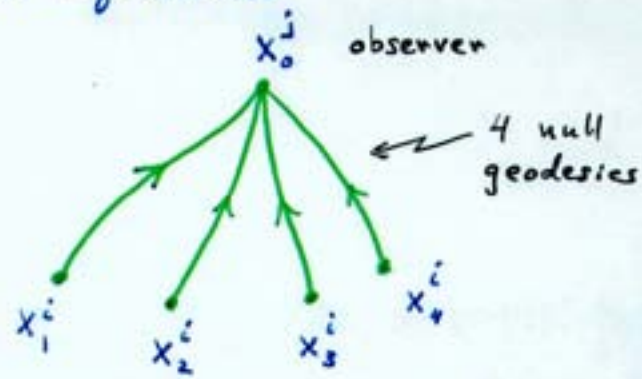
Example: Minkowski flat space

$$\Omega(x_1^i, x_2^j) = \frac{1}{2}\eta_{ij}(x_2^i - x_1^i)(x_2^j - x_1^j)$$
$$= \frac{1}{2}\left[\sum_{i=1}^{3}(\Delta x^i)^2 - (c\,\Delta t)^2\right]$$

## General Relativistic Navigation Equations

$$\boxed{\Omega(x_a^i, x_o^j) = 0 \qquad a = 1,2,3,4}$$

covariant & invariant




radio beacons or satellite emission events

<u>In terms of GPS observables</u>

$$\Omega(t_1, \vec{x}_1, t_2, \vec{x}_2) = \tilde{\Omega}(t_2 - t_1, \vec{x}_1, \vec{x}_2)$$

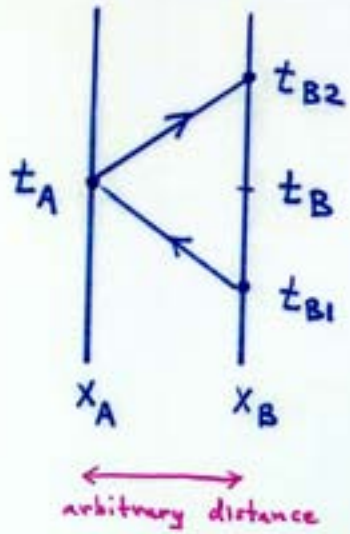
$$\tilde{\Omega}(\overbrace{\Delta t_a + \delta t_o^*}^{t_o - t_s}, \vec{x}_a, \vec{x}_o) = 0, \quad a = 1,2,3,4 \text{ satellites}$$

measured pseudorange — ? observer clock offset — broadcast ephemeris — ? observer position

Note: For time transfe to a known spatial location, $\vec{x}_o$, only o satellite is needed.

# Special Relativity: Clock Synchronization by Exchange of Light Signals

Clock A and Clock B are at rest in RF. Synchronize Clock B to Clock A.




## Definition of Simultaneity

Event at $x_B$ at time

$$t_B = \frac{1}{2}(t_{B1} + t_{B2}) \qquad (1)$$

$$= t_{B1} + \frac{1}{2}(t_{B2} - t_{B1})$$

is simultaneous with event at $x_A$ at $t_A$

## Synchronization – Operational Definition

[Method for constructing grid of synchronized coordinate clocks.]

Clock B will be synchronized with clock A, if at time $t_{B2}$ clock B reads

$$t_{B2} = t_A + (t_{B2} - t_B) \quad \text{(definition)}$$

use def. in Eq. (1)

$$t_{B2} = t_A + \frac{1}{2}\underbrace{(t_{B2} - t_{B1})} \quad \text{(expressed in measurable quantities)}$$

From reflection of clock A face seen at $x_B$ at $t_{B2}$

Elapsed time between emission and reception, measured by clock B

$$t_B = t_A$$

Simultaneous events occur at same coordinate time

[ Note: $t_{B2} = t_A + \dfrac{|x_A - x_B|}{c}$ ]

NOTE: Speed of light c does not enter explicitly.

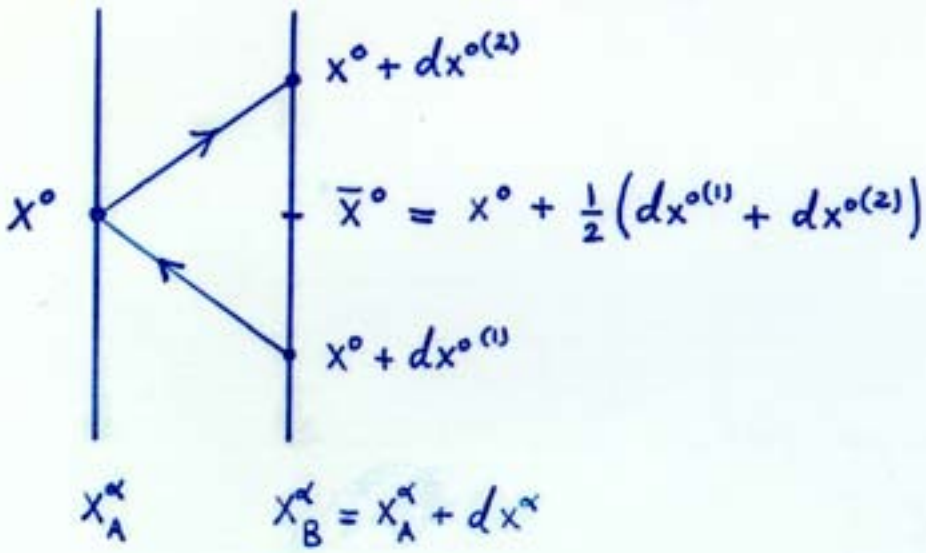
NOTE: Only synchronization was performed – not syntonization, (rate adjustment) so assumed clocks run at same rate.

# General Relativity : Clock Synchronization by Exchange of Light Signals

Landau and Lifshitz

Coordinates: $x^0 = ct$, $x^\alpha = (x^1, x^2, x^3)$

## Definition of Simultaneity




The midpoint $\bar{x}^0$ is defined to be simultaneous with $(x^0, x^\alpha_A)$

For light propagation: $0 = ds^2 = g_{00}(dx^0)^2 + 2g_{0\alpha}dx^0dx^\alpha + g_{\alpha\beta}dx^\alpha dx^\beta$

$\alpha, \beta = 1, 2, 3$

Solve quadratic for $dx^0$

$$dx^{0(1,2)} = \frac{1}{g_{00}}\left[-g_{0\alpha}dx^\alpha \pm \left(g_{0\alpha}g_{0\beta} - g_{00}g_{\alpha\beta}\right)dx^\alpha dx^\beta\right]$$

Coordinate times $dx^{0(1)}$ and $dx^{0(2)}$ for light propagation depend on direction of propagation.

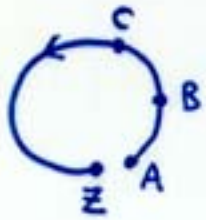
Then: $$\boxed{\bar{x}^0 = x^0 - \frac{g_{0\alpha}}{g_{00}}dx^\alpha}$$

Simultaneous events occur at <u>different</u> values of coordinate time!

- Special Relativity $g_{ij} = \eta_{ij} \Rightarrow g_{0\alpha} = 0$ and $\bar{x}^0 = x^0$.
- In GR $\bar{x}^0 \neq x^0$ is a result of choice of coordinate system (actually RF), not the space. We can always choose RF where $g_{0\alpha} = 0$.

# General Relativity: Synchronization of Clocks

Assume metric $g_{ik}$ is time-independent




Coordinate time difference for simultaneous events: A, B, C, ... Z

$$\Delta x^0 = -\oint \frac{g_{0\alpha}}{g_{00}} dx^\alpha \neq 0 \quad \text{in general}$$

where $\alpha = 1,2,3$

- Synchronization of clocks is path dependent!
  (unless $g_{0\alpha} = 0$, or $\frac{g_{0\alpha}}{g_{00}} = \nabla\phi$)

- If $\Delta x^0 \neq 0$, synchronization over all space is impossible.

- Proper time elapsed on real clock, $d\tau$, as compared to coordinate time $dx^0 = d(ct)$, depends on position of clock:

$$ds^2 = (c\,d\tau)^2 = g_{ij}\,dx^i\,dx^j$$

$$= g_{00}(dx^0)^2 + 2g_{0\alpha}\,dx^0 dx^\alpha + g_{\alpha\beta}\,dx^\alpha dx^\beta$$

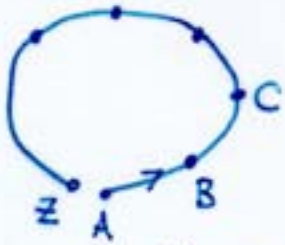
$\alpha, \beta = 1,2,3$

Stationary Clock: $dx^\alpha = 0$

$$\therefore \underbrace{c\,d\tau}_{\text{time elapsed on real clock}} = \underbrace{\sqrt{g_{00}}}_{\text{function of position}}\; \underbrace{dx^0}_{= d(ct) = \text{elapsed coordinate time}}$$

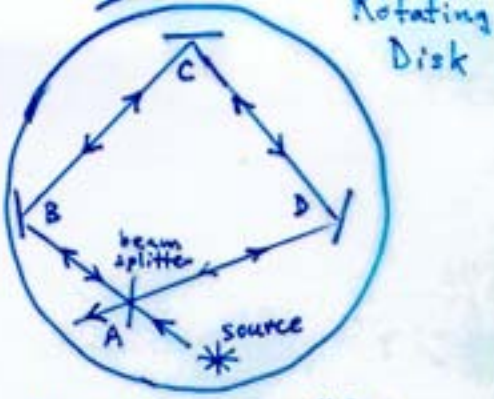
# Sagnac Effect

## Clock Synchronization is Path Dependent




Events A, B, C, ... Z are simultaneous.

$$t_Z - t_A = -\frac{1}{c}\oint \frac{g_{0\alpha}}{g_{00}}\,dx^\alpha, \quad \alpha = 1,2,3$$




Fringe shift $\Delta N = \frac{4S\omega}{c\lambda}$

$S$ = area enclosed

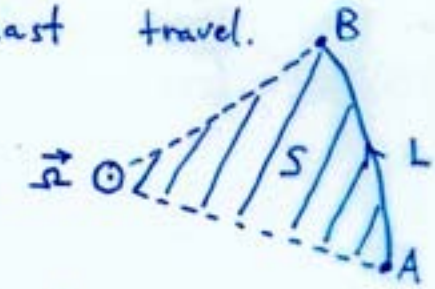
(Sagnac 1915)

1. Clocks cannot be (Einstein) synchronized on the rotating earth, i.e., in Earth-Centered Earth Fixed Frame

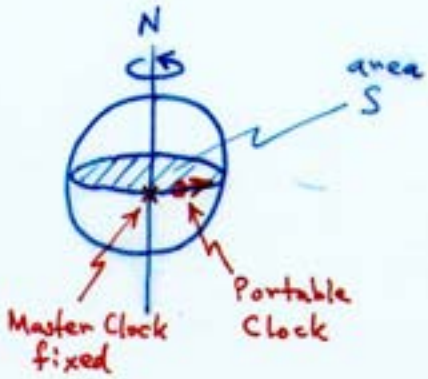
2. Simultaneous events in space-time occur at <u>different</u> values of coordinate time $t$, when $\frac{g_{0\alpha}}{g_{00}} \neq \nabla\varphi$.

3. Coordinate time for light travel around the earth is different for: east to west than west to east travel.

$$\Delta t_\pm = \frac{L}{c} \pm \underbrace{\frac{2\,\vec{\Omega}\cdot\vec{S}}{c^2}}_{\substack{207.3\text{ ns} \\ \text{for } 360^\circ \text{ on equator}}}$$




4. Portable Clock Transport.
After one round trip eastward a portable clock will lag behind a stationary Master clock by 207.3 ns.



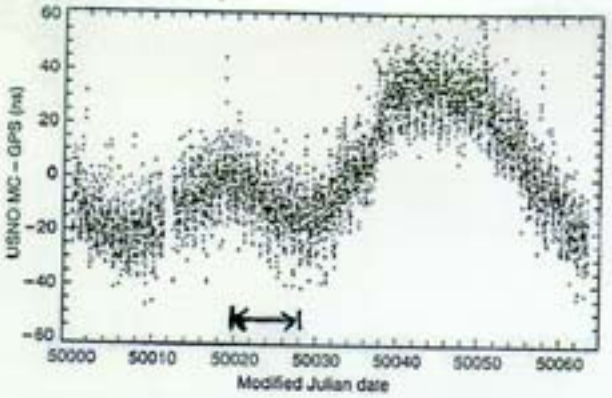



Fig. 1. Original data (USNO Master Clock)—GPS time in nanoseconds versus the MJD for period MJD 50000.00553 to 50063.54009.

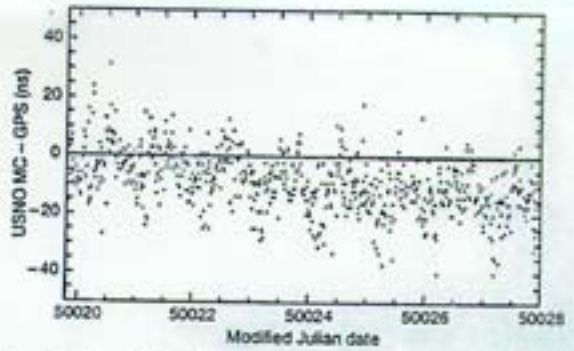



Fig. 2. Portion of data from MJD 50020 to 50028 on expanded scale.

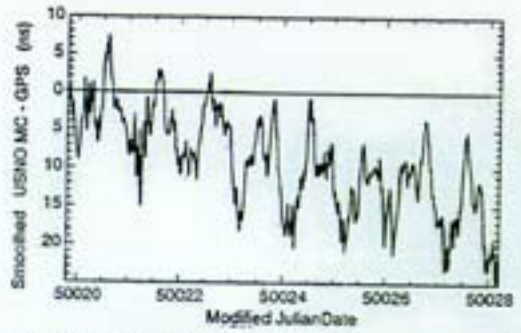



Fig. 3. Original data, smoothed by taking average of eleven points, for time period MJD 50020 to 50028. Diurnal oscillation of (peak-to-peak) magnitude of 18 ns to 20 ns is evident.

the Kalman filter process run at the Master Control Station in Colorado Springs and the response of other sub-system components. There is a gap in the data during the period MJD 50011.45177 to 50012.46282, due to data aquisition difficulties.

Fig. 2 shows a subset of the same data over the time period MJD 50020 to 50028. A salient feature of the data is the scatter of points, on the order of 30 ns peak-to-peak. This scatter is due to a combination of effects [1], including noise in the receiver electronics ($\sim 0.5$ m $\approx 1.5$ ns), multipath effects ($\sim 1.4$ m), tropospheric delay ($\sim 0.7$ m), uncompensated ionospheric delay ($\sim 1.2$ m), satellite clock errors ($\sim 2.1$ m), and ephemeris errors ($\sim 2.1$ m), where all values are 1 $\sigma$. The USNO GPS antenna phase center coordinates are believed accurate to approximately 0.5 m. Perhaps the biggest contribution to this scatter in the data is from the fact that all satellites in the constellation do not have their navigation message updated at the same time, resulting in a scatter of values during any given measurement time. Besides this scatter, a diurnal variation is apparent. Fig. 3 shows an eleven point average of the data in the time period MJD 50020 to 50028. A diurnal oscillation of magnitude 18 ns to 20 ns peak-to-peak is clearly present. The physical origin of this diurnal variation is not well understood. Workers in the field [18] have proposed a variety of reasons to explain this behavior, including broadcast ephemeris errors, multipath errors, incorrect receiver antenna phase center coordinates, the fundamental accuracy limit of the clocks on the Block II satellite vehicles, poor thermal control of the clock systems (on the ground and in the satellites), inaccuracies in modeling of the ionosphere and troposphere, and that relativistic effects in the GPS have not been accounted for properly.

### A. Fourier Transform

We search for periodicity in the data by performing a Fourier transform using a fast Fourier transform (FFT) algorithm. The FFT algorithm [23] requires that the data set be uniformly sampled in time, however, the original data set is not uniformly sampled. Therefore we fit a cubic spline to the original data set and resample the data at a uniform sampling rate $\Delta = t_{i+1} - t_i = 0.002$ day, where $t_i$ is the time of the $i$th resampled data point. The sampling rate of the original data varied in the approximate range 0.009 to 0.013 day, so we have lost essentially no information by resampling the data using a smaller sampling interval. For the Fourier transform $H(f)$ of a function $h(t)$ we use the convention

$$H(f_n) = \int_{-\infty}^{+\infty} h(t) e^{2\pi i f_n t}\, dt \approx \Delta \sum_{k=0}^{N-1} h_k e^{2\pi i f_n t_k}$$

$$= \Delta \sum_{k=0}^{N-1} h_k e^{2\pi i k n/N} = \Delta H_n \tag{3}$$

where $h_k = h(t_k)$. For $N$ data points there are $N$ Fourier amplitudes at frequencies $f_n$, where $n = -N/2, \ldots, 0, \ldots, +N/2$. The finite data set imposes the periodicity on the amplitudes $H_{n+N} = H_n$. In particular, the amplitudes $H_{-N/2}$ and $H_{N/2}$ are equal and not independent. Positive frequencies $0 < f < f_c$, where $f_c = 1/(2\Delta)$ is the Nyquist critical frequency, correspond to discrete $f_n$ for $n = 1, 2, \ldots, N/2 - 1$. Furthermore, our data consist of a real function so the amplitudes for negative frequencies are related to the amplitudes for positive frequencies $H(-f) = H(f)^*$, or in terms of the discrete amplitudes, $H_{-n} = H_n^*$.